Forces and work on a wire in a magnetic field

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(Received 9 February 2011; accepted 28 March 2011)

We illustrate the role of magnetic forces in lifting a current-carrying wire by discussing the various forces acting on the positive ions and the electrons that compose the wire.

[DOI: 10.1119/1.3579422]

I. INTRODUCTION

Classically, a static magnetic field $\vec{B}$ does no work because the magnetic force $\vec{f}_{\rm mag}=q\vec{v}\times\vec{B}$ on a particle of charge q is always perpendicular to its velocity $\vec{v}$. This property can be extended to the magnetic force acting on a macroscopic number of particles, such as the conduction electrons in a current-carrying wire. This remarkable property of the magnetic force gives rise to a number of apparent paradoxes.[1–3]

An interesting example is the acceleration of atoms by the magnetic field in a Stern–Gerlach experiment.[4] In this case, the atoms, with (orbital) magnetic moments, accelerate in the nonuniform magnetic field at the expense of their internal kinetic energies. For the electron-spin contribution to the magnetic moments, the situation is not as clear. If the magnetic field does no work, then the electron would need to have an intrinsic rotational kinetic energy associated with its spin.[4] Another example[1,2] is that of the electromotive force (emf) that appears in a loop of wire moving (with constant velocity $\vec{V}$) in a region in which there is a static magnetic field. Even though this emf is given by an integral of the tangential magnetic force per unit charge $f_{{\rm mag},t}/q=VB$, the work can be attributed to the external agent that moves the circuit. The magnetic force does no work, as expected.

Griffiths[2] discussed an example in which the perpendicular magnetic force $F_{{\rm mag},\perp}=IaB$ lifts a rectangular loop of wire with sides a and b in which there is a constant current I. A rectangular loop of wire of mass m_w hangs vertically with one side in a constant and uniform magnetic field $\vec{B}$, which points into the page in the region above the dashed horizontal line (see Fig. 1). An electric current, generated by a power supply not shown in the figure, circulates clockwise.

We can determine the current I_0 for which the upward magnetic force in the loop ($\vec{F}_{\rm mag}=I\vec{a}\times\vec{B}$) (see Fig. 1 for the definition of $\vec{a}$) exactly balances the weight downward: $I_0=m_w g/Ba$. We can analyze what happens at a higher constant current $I>I_0$. As pointed out in Ref. 2, we expect that the magnetic force exceeds the weight and, thus, the loop rises. It is argued that if the loop rises a distance h, we would be tempted to write for the work of the vertical magnetic force that lifts the loop,

$$W_{\rm mag}=F_{\rm mag}h=IBah, \tag{1}$$

which would contradict the fact that magnetic forces do no work.

This apparent paradox arises because the magnetic force on a moving current-carrying wire in a static and uniform magnetic field $\vec{B}$ is not given by $\vec{F}_{\rm mag}=I\vec{a}\times\vec{B}$. We have to recognize that the wire is a superposition of positive (ion) and negative (electron) charge distributions, each of which feels a different magnetic force. These two charge distributions also interact through electric forces, for which there is no constraint on the mechanical work. McKinnon *et al.*[5] called the electric force $\vec{F}$ on the positive ions making up the lattice the force on the wire and showed that $\vec{F}=I\vec{a}\times\vec{B}$. Here, we provide an example (the rectangular loop of wire) to emphasize that this equality is not always true. We also discuss the mechanical work due to the different forces that act on this system. The discussion of this example in terms of the macroscopic forces on the loops of positive ions and electrons might help give students a sense of the unity of classical electromagnetism and mechanics and help them understand the issue of work and magnetic forces.

II. RESULTS

We model the current-carrying loop of wire as a superposition of a rectangular rigid loop of mass m with moving conduction electrons and a rectangular rigid loop of mass M containing the lattice of positive ions. The two loops rise together along the y axis with velocity $v(t)$ and, at the same time, the conduction electrons are circulating around the loop with the constant drift velocity v_d associated with the current I. Such a decomposition of a wire into positive and negative charge distributions can be found in literature, such as in the derivation of the magnetic field of a current-carrying wire using relativistic concepts (see, for example, Ref. 6).

Two external agents exert forces on these two loops of charges: The magnetic field and the power supply (which pulls the electrons along the circuit). Our simplifying assumption of a constant electric current I means that the power supply has to be a current source, and not a voltage source (such as a battery), which would produce a circuit of constant emf, and a variable current. The two cases are similar, but for a constant I the loops rise with constant acceleration, whereas for a constant emf the acceleration is variable and the velocity of the loops attains a constant limiting value.

In Fig. 2, we show the free-body diagrams representing the forces acting on these two loops (the loops lie only partly in the magnetic field). The loop of positive ions experiences the transverse force $\vec{F}$ transmitted to it by the conduction electrons,[1,5,7] the horizontal magnetic force $\vec{F}^{(+)}_{\rm mag}$, and the weight $M\vec{g}$. In the loop of moving electrons, the electrons experience the magnetic force $\vec{F}^{(-)}_{\rm mag}=\lambda a v(t)B\hat{x}+IaB\hat{y}$, the reaction $-\vec{F}$, and the weight $m\vec{g}$ (λ is the linear charge density of the conduction electrons, which we assume is uniform). The transverse force $\vec{F}$ and its reaction are electrical in na-

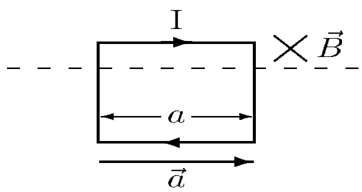

Fig. 1. A rectangular loop of wire hanging vertically in a uniform magnetic field (which exists only above the dashed line). The length vector $\vec{a}$ is shown.

ture. They originate from the Hall field created by surface charges on the wire, which are induced by the magnetic forces on the conduction electrons.[1,5,7]

Forces that act tangential to these loops [other than $F^{(-)}_{\mathrm{mag},x}=\lambda a v(t)B$], such as the dissipative resistive force (friction) on the conduction electrons $\vec{F}_r$ and the force from the power supply that drives these electrons $\vec{F}_{\mathrm{ps}}$, and its reactions are not shown in these diagrams. [We neglect $\vec{F}_r$ here because it is canceled by $\vec{F}_{\mathrm{ps}}$ along the circuit, but the existence of this force is implicit in the term RI of Eq. (10).] The force $\vec{F}_{\mathrm{ps}}$ is transmitted to the conduction electrons through surface charge densities along the surface of the wire. (These charges, as well as those that are responsible for the Hall field, are attached to the loop of positive ions.) This mechanism for the behavior of circuits in terms of surface charges is discussed in Ref. 8. Because there is no horizontal movement of the loop of electrons, the power supply has to exert on this loop a resultant force in the direction of $-x$. This force is responsible for maintaining the current against the tangential magnetic force $F^{(-)}_{\mathrm{mag},x}$. The reaction to this resultant force acts on the loop of positive ions, canceling with $\vec{F}^{(+)}_{\mathrm{mag}}$. Therefore, as expected, the resultant force on the wire (ions+electrons) is $(F^{(-)}_{\mathrm{mag},y}-(M+m)g)\hat{y}$.

We can show using these force diagrams that it is incorrect to think, in general, that the force $\vec{F}$ communicated to the positive lattice is equal to the transverse magnetic force $\vec{F}^{(-)}_{\mathrm{mag},y}=IaB\hat{y}$.[5] Applying Newton's second law in component form (along y) to the loop of ions and to the loop of conduction electrons yields

$$F-Mg=Ma_y \tag{2}$$

and $F^{(-)}_{\mathrm{mag},y}-F-mg=ma_y$, which implies that

$$IaB-F-mg=ma_y. \tag{3}$$

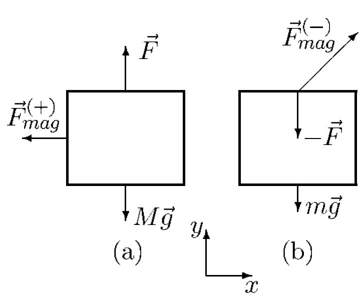

Fig. 2. Free-body diagrams representing forces acting on (a) the loop of positive ions and (b) the loop of conduction electrons. The loops are moving together along the y axis and lie only partly in the magnetic field.

From Eqs. (2) and (3), we obtain

$$F=\frac{IaB}{1+m/M}, \tag{4}$$

which shows that the transverse force on the loop of positive ions is not usually equal to IaB. (The equality is valid only if we can neglect the ratio m/M, as done in Ref. 5.) In practice, we expect that the ratio m/M is small and can be neglected. We can estimate the value of this ratio using the data for copper: The density of mobile charge carriers per unit volume is $n\approx 8.5\times 10^{28}\ \mathrm{m}^{-3}$ and the mass density is $\rho\approx 8930\ \mathrm{kg/m}^3$, which gives $m/M\approx 9\times 10^{-6}$.

For the vertical velocity $v(t)$ of the loop of wire, we obtain

$$v(t)=v(0)+\left(\frac{IaB}{M+m}-g\right)t. \tag{5}$$

We now investigate the energetics of the system, assuming that the loops rise a height Δh in the time interval Δt. For the loop of positive ions, there is no magnetic force along y, and the balance between work and energy gives

$$\Delta E_{\mathrm{ions}}=\Delta K_{\mathrm{ions}}+\Delta U_{\mathrm{ions}}=W_F=\frac{M}{M+m}IaB\Delta h, \tag{6}$$

where K is the kinetic energy and U is the gravitational potential energy.

For the loop of conduction electrons,

$$\Delta E_e=\Delta K_e+\Delta U_e=W_{F^{(-)}_{\mathrm{mag},y}}-W_F=\frac{m}{M+m}IaB\Delta h. \tag{7}$$

While the electrons circulate against the force $F^{(-)}_{\mathrm{mag},x}$, this force does work given by (using $dx=v_d dt$ and $I=\lambda v_d$)

$$W_{F^{(-)}_{\mathrm{mag},x}}=-\int_{\Delta t}\lambda a v(t)Bv_d dt=-IaB\Delta h. \tag{8}$$

Therefore, from Eqs. (6)–(8), we can confirm that

$$W_{F^{(-)}_{\mathrm{mag}}}=W_{F^{(-)}_{\mathrm{mag},x}}+W_{F^{(-)}_{\mathrm{mag},y}}=0. \tag{9}$$

Because the magnetic field does no work on the rising loops, the gain in energy, $\Delta E_{\mathrm{ions}}+\Delta E_e$, is due to the other external agent, which also acts on the system: The power supply. To calculate the work of this agent, we invoke the circuit laws

$$\epsilon(t)-av(t)B=RI, \tag{10}$$

which is Ohm's law, where $\epsilon(t)$ is the time-dependent emf which the power supply must impose on the circuit to keep constant the current ($\epsilon(t)=RI+av(t)B$) and R is the electrical resistance of the wire. Also, the electric power delivered to the wire by the power supply is $P(t)=I\epsilon(t)$. Therefore, the work done by the power supply is

$$W_{\mathrm{ps}}=\int_{\Delta t}I\epsilon(t)dt=RI^2\Delta t+IaB\Delta h, \tag{11}$$

which shows that $W_{\mathrm{ps}}=Q+\Delta E_{\mathrm{ions}}+\Delta E_e$, where $Q=RI^2\Delta t$ is the energy generated through Joule heating during the time Δt. This result confirms that the mechanical energy gained by the two rising loops and the energy lost through Joule heating are provided by the power supply connected to the circuit.

What is the role of the magnetic forces in the rising loop of wire? As discussed in Ref. 2, the role of the magnetic forces here is to redirect the horizontal force of the power supply into the vertical motion of the loops of charge. This mechanism of redirection is accomplished through the electric force F (which lifts the ions) and originates in the surface charges induced on the wire by the magnetic forces.

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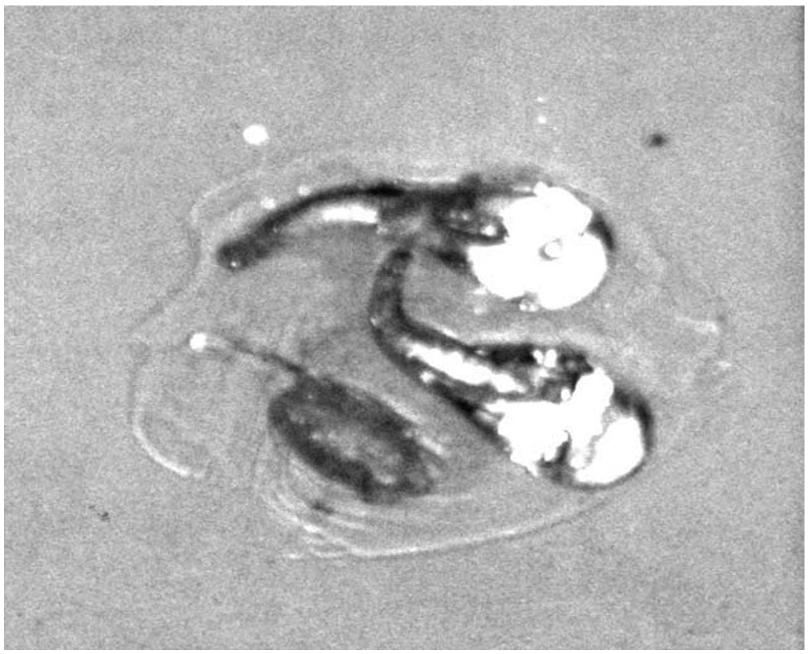

Prince Ruperts Drops. Price Rupert's drops are produced by melting small amounts of glass from the end of a rod, and letting the resulting teardrop- shaped blobs of glass fall into a vat of water. In the process of rapid quenching, tremendous internal strains are set up. The stress patterns in the glass can be observed by placing it between crossed polarizing filters. Breaking the end from a drop will cause it to shatter into tiny pieces. Beware: this is a dangerous demonstration! These drops, in the collection of the United States Military Academy at West Point, have been glued to a thin sheet of glass for use in a projector. See: Clifton Albergotti, ″Prince Rupert's Drops in Literature″, Phys. Teach., 27, 530-532 (1989) (Notes and photograph by Thomas B. Greenslade, Jr., Kenyon College)